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Banach Contraction Principle in Generalized Non Archimedean Menger PM Space

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Abstract

The aim of this paper is to introduce the concept of generalized non-Archimedean Menger PM-spaces and to obtain Banach contraction principle in this newly defined space. We also provide an example in support of our result.

Keywords: *Banach Contraction Principle, Non Archimedean Menger PM Space.*

Introduction

In recent years many great developments have been made in the theory and applications of metric spaces, 2-metric spaces, D-metric spaces and probabilistic metric spaces. Recently, Chang and Huang [2] defined the concept of probabilistic 2-metric spaces and studied their properties. In 1999, Chugh and Sumitra their basic properties. Motivated by the concepts of 2-metric spaces, D-metric spaces (generalized metric spaces), probabilistic 2-metric spaces and 2-non-Archimedean Menger PM-spaces (briefly GNA Menger PM-spaces).

Chang and Huang [2] introduced the concept of probabilistic 2-metric spaces as follows:

Definition 1.1. A probabilistic 2-metric space is a pair (X, F) where X is non-empty set and F is a mapping from $X \times X \times X$ into L satisfying the following conditions (for u, v, w $\in X \times X \times X$, the distribution function is represented by F(u, v, w) or $F_{u, v, w}$ and the value of F(u, v, w) at t $\in$ R by F(u, v, w; t) or $F_{u, v, w}(t)$ for all u, v, w $\in$ X and t > 0).

(i) F(u, v, w; 0) = 0

(ii) for distinct u, v, in X, there exists a point w in X such that F(u, v, w; t_o) < 1 for some t > 0

(iii) F(u, v, w; t) = 1 if and only if at least two of them are equal

(iv) F(u, v, w; t) = F(u, w, v; t) = F(v, w, u; t) (Symmetry)

(v) If F(u, v, s; t_1) = F(u, s, w; t_2) = F(s, v, w; t_3) = 1, then F (u, v, w; ($t_1 + t_2 + t_3$)) = 1.

Definition 1.2 A mapping t : $[0, 1] \times [0, 1] \times [0, 1] \to [0, 1]$ is a T-norm t if it is associative, commutative, non-decreasing in each co-ordinate and t(a, 1, 1) = a for every a $\in$ [0, 1].

Definition 1.3. A probabilistic 2-Menger PM-space is an order triple (X, F, t), where t is a T-norm and (X,F) is a probabilistic 2-metric space satisfying the following condition:

(vi) F(u, v, w; ($t_1 + t_2 + t_3$)) $\geq$ t(F(u,v,s; t_1), F(u, s, w; t_2), F(s, v, w; t_3)) for all u, v, w $\in$ X and t_1, t_2, $t_3 \in R^+$.

Chugh and Sumitra [1] introduced the concept of 2-non-Archimedean Menger PM-space as follows:

Definition 1.4. Let X be any non-empty set and L be set of all left-continuous distribution functions. An ordered pair (X, F) is said to be a 2-non-Archimedean PM-space (briefly 2-N.A.

PM-space) if F is mapping from X × X × X into L with properties (i), (ii), (iii) and (iv) of the definition 1.1. and (v) is replaced by (v)

(v) if $F(u, v, s; t_1) = F(u, s, w; t_2) = F(s, v, w; t_3) = 1$, then

$$F(u, v, w; (\max(t_2, t_2, t_3)))=1$$

Definition 1.5. A 2-non-Archimedean Menger PM-space is a triple (X, F, t), where (X, F) is a 2-N.A. PM-space and t is a T-norm with the following condition:

(vi) $F(u, v, w; (\max(t_1, t_2, t_3)) \geq t(F(u, v, s; t_1), F(u, s, w; t_2), F(s, v, w; t_3))$

for all $u, v, w \in X$ and $t_1, t_2, t_3 \geq 0$.

The concept of 2-metric space has been investigated by S. Gahlar [8] and has been developed extensively by many mathematicians.

Definition 1.6. A 2-metric space is defined by a function $d : X \times X \times X \to R$ with the following properties

(1) for each distinct pair u, v in X there exists a w ε X such that $d(u, v, w) \neq 0$

(2) $d(u, v, w) = 0$ when at least two of the points are equal

(3) $d(u, v, w) = d(v, u, w) = d(u, w, v) =$ .... (symmetry)

(4) $d(u, v, w) \leq d(u, v, a) + d(u, w, a) + d(a, v, w)$ for all a in X. It is easily seen that d is non-negative.

A number of fixed point theorems have been proved for 2-metric spaces. However, Hsiao [8] showed that all such theorems are trivial in the sense that the iterations of mappings are collinear.

In 1984, Dhage [4] introduced the concept of D-metric space (generalized metric space) as follows:

Definition 1.7. Let $D: X \times X \times X \to R$ with the properties (1), (3), (4) of definition 1.6 (2) is replaced by (2)′.

(2)′ $D(u, v, w) = 0$ if and only if $u = v = w$.

Dhage introduced the concept of D-metric space by replacing (2) by (2)′. Motivated by the concepts of 2-metric spaces, D-metric spaces (generalized metric spaces) probabilistic 2-metric spaces and 2-non-Archimedion Menger PM spaces, we introduce the concept of generalized Non-Achimendean Menger PM-spaces (briefly GNA Menger PM-spaces) as follows:

Definition 1.8. Let X be a non-empty set and L be set of all left continuous distribution functions. An ordered pair (X, F) is said to be a generalized non-Archimedean PM-space (briefly GNA PM-space) if F is a mapping from $X \times X \times X \to$ into L with the properties (i), (ii), (iv) of definition 1.1 and (v)′ of definition 1.4 and (iii) is replaced by (iii)′

Definition 1.9. A generalized non-Archimedean Menger PM-space (briefly GNA Menger PM-space) is a triple (X, F, t), where (X, F) is a GNA Menger PM-space and t is a T-norm with the property (vi) of definition (1.5).

The concept of neighbourhoods in GNA Menger PM-space is as follows:

If u ε X, $\in > 0$, λ ε (0, 1) then an $(\in, \lambda)$-neighbourhoods of u, $U_u(\in, \lambda)$ is defined by $U_u(\in, \lambda) = \{v, w, \varepsilon X: F(u, v, w; \in) > 1 - \lambda\}$.

If (X, F, t) is a GNA Menger PM-space with continuous T-norm t then the family $\{U_u(\in, \lambda) : u \ \varepsilon\ X, \in > 0, \lambda\ \varepsilon\ (0, 1)\}$ of neighbourhoods induce a Hausdorff topology on X if $\sup_{a<1} t(a, a, a) = 1$.

An important T-norm t is $t(a, b, c) = \min \{a, b, c\}$ for all $a, b, c \in [0, 1]$ and this is the unique T-norm t such that $t(a, a, a) \geq a$ for every a ε [0, 1]. Indeed, if it satisfies this condition, we have

$\min \{a, b, c\} \leq t(\min \{a, b, c\}, \min \{a, b, c\}, \min \{a, b, c\})$

$\leq t(a, b, c) \leq t(\min\{a, b, c\}, 1, 1) = \min \{a, b, c\})$ therefore t = min.

Definition 1.10. A GNA PM-space (X, F) is said to be of type $(C)_g$ if there exists a $g \in \Omega$ such that

$g(F(u, v, w; t) \leq g(F(u, v, s; t)) + g(F(u, s, w; t)) + g(F(s, v, w; t))$ for all $u, v, w, s \in X$ and $t \geq 0$, where $\Omega = \{g : g : [0, 1] \to [0, \infty)$ is continuous, strictly decreasing, $g(1) = 0$ and $g(0) < \infty\}$.

Definition 1.11. A GNA Menger PM-space (X, F, t) is said to be of type $(D)_g$ if there exists a $g \in \Omega$ such that $g(t(r, s, t) \leq g(r) + g(s) + g(t)$ for all r, s, t ε [0, 1].

Remark 1. If a GMA Menger PM space (X, F, t) is of type $(D)_g$, then (X, F, t) is of type $(C)_g$.

2. If (X, F, t) is a GNA Menger PM-space and $t \geq t_m$, where $t_m (r, s, t) = \max \{r + + t -1, 0,$

0} then (X, F, t) is of type $(D)_g$ for $g \in \Omega$ defined by g(t) = 1 – t.

Definition 1.12. A sequence $\{x_n\}$ in GNA Menger PM-space is called a convergent sequence if $\lim_{m,n} g(F(x_n, x_m, x_p; t)) = 0$.

Definition 1.13. A sequence $\{x_n\}$ in GNA Menger PM-space is called a Cauchy sequence if $\lim_{m,n,p} g(F(x_n, x_m, x_p; t)) = 0$.

Definition 1.14. A complete generalized non-Archmedian Menger PM-space X is one in which every cauchy sequence $\{x_n\}$ in X converges to a point x in X.

Throughout this paper, let (X, F, t) be a complete GNA Menger PM-space of type $(D)_g$ with a continuous strictly increasing T-norm t.

Let $\phi : [0, \infty) \to [0, \infty)$ be a function satisfying the following condition (Φ).

(Φ) φ is upper semicontinuous from the right and φ(t) < t for all t > 0.

Lemma 1.1 [3]. If a function $\phi : [0, \infty) \to [0, \infty)$ satisfies the condition (Φ) then we have

(1) For all $t \geq 0$, $\lim_{n\to\infty} t_n = 0$, where $\phi^n(t)$ is the n-th iteration of φ(t).

(2) If $\{t_n\}$ is non-decreasing sequence of real numbers and $t_{n+1} \leq \phi(t_n)$, n = 1, 2, ..., then if $t \leq \lim_{n\to\infty} t_n = 0$. In particular, if $t \leq \phi(t)$ for all $t \geq 0$, then = 0. Now we prove Banach Contraction principle in this newly defined space as follows:

2. Main Result

Theorem 2.1. Let f be a continuous self mapping of a complete generalized non-Archimedean Menger PM-space (X, F, t) where t is a continuous T-norm such that

$$g(F(fx, fy, fz; t)) \leq \phi(g(F(x, y, z; t))) \quad (2.1)$$

for all x, y, z in X and t > 0. Then f has a unique fixed point in X.

Proof. Let x_0 be an arbitrary point of X. Construct a sequence $\{x_n\}$ in X defined by $x_{n+1} = fx_n = 0, 1, 2, \ldots$.

If $x_n = x_{n+1}$ for some n then x_n is a fixed point of f. So we can assume $x_n \neq x_{n+1}$ for every n. Put $x = x_n$, $y = x_{n+1}$ and $z = x_p$ in (2.1), we have $g(F(x_n, x_{n+1}, x_p; t)) = g(F(fx_{n-1}, fx_n, fx_{p-1}; t))$

$$\leq \phi(g(F(x_{n-1}, x_n, x_{p-1}; t))$$

Proceeding limit as $n \to \infty$, Lemma 1.1 yields, $g(F(x_n, x_{n+1}, x_{n+p}; t)) = 0$. Now for any position integer p and t,

$$\begin{aligned} g(F(x_n, x_{n+p}, x_{n+p+t}; t)) &\leq g(F(x_n, x_{n+1}, x_{n+p+t}; t) + g(F(x_n, x_{n+1}, x_{n+p}; t) \\ &\quad + g(F(x_{n+1}, x_{n+p}, x_{n+p+t}; t) \\ &\leq g(F(x_n, x_{n+1}, x_{n+p+t}; t) \\ &\quad + g(F(x_n, x_{n+1}, x_{n+p}; t) + g(F(x_{n+1}, x_{n+2}, x_{n+p+t}; t) \\ &\quad + g(F(x_{n+1}, x_{n+2}, x_{n+p}; t) + g(F(x_{n+2}, x_{n+p}, x_{n+p+t}; t) \\ &\quad + \ldots \end{aligned}$$

$$g(F(x_n, x_{n+p}, x_{n+p+t}; t)) \leq \sum_{k=n}^{n+p+t} g(F(x_k, x_{k+1}, x_{n+p+t}; t)) + \sum_{k=n}^{n+p} g(F(x_k, x_{k+t}, x_{n+p}; t))$$

Proceeding limit as $n \to \infty$, we have $\lim_{n\to\infty} g(F(x_n, x_{n+p}, x_{n+p+t}; t) = 0$. Therefore, $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists a point x ε X such that $\lim_{n\to\infty} x_n = x$.

Also f is continuous. Therefore, $x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} fx_n = f(\lim_{n\to\infty} x_n) = fx$ i.e., x is a fixed point of f, then

$$g(F(x, x, y; t)) = g(F(fx, fx, fy; t) \leq \phi(g(F(x, x, y; t))$$

Thus by Lemma 1.1., we have x = y. This completes the proof.

Example. Let X = [0, 1] and D: X× X× X → R, defined by

$$D(x, y, z) = \begin{cases} 0 & \text{if } x = y = z \\ 1 & \text{if } x \neq y \neq z \end{cases}$$

Then (X, D) is a complete D-metric space, (see Dhage [5]).

Set F(x, y, z; t) = H(t – D(x, y, z)), t > 1

where $F(x, y, z; t) = \begin{cases} H(t) & \text{for } x \neq y \neq z \\ 1 & \text{for } x = y = z \end{cases}$

and

$$H(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ 1 & \text{if } t > 0 \end{cases} \quad \text{with}$$

t(a, b, c) = min (a, b, c). Then clearly (X, F, t) is a complete generalized non-Archimedean Menger PM-space. Set f(x) = 1 for all x ε X, then all the conditions of Theorem 2.1 are satisfied and 1 is the fixed point of f.

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